JOURNAL OF APPROXIMATION THEORY 18, 388–389 (1976)

Letter to the Editor

On the Unicity in Simultaneous Approximation by Algebraic Polynomials

The problem of uniqueness of best approximating polynomial to $f \in C^{k_p+1}[a, b]$ from Π_n with respect to the norm: $\| f \|_F = \max_{k_i \in F} \| f^{(k_i)} \|$, where $F = \{0 = k_0 < k_1 < \cdots < k_p \leqslant n\}$ and $\| \cdot \|$ is the supremum norm on $[a, b]$, is treated in [3]. In this note it is shown that the main result of [3] can be derived directly from the unicity in Restricted Derivatives Approximation (RDA) [1, 4].

The following result is proved in [3, Theorem 4].

THEOREM. *Let $f \in C^{k_p+1}[a, b]$ and let $\Omega(f)$ be the set of all polynomials of best approximation to $f(x)$ from Π_n in the norm $\| \cdot \|_F$. If, for all $p \in \Omega(f)$,*

$$\| f - p \|_F = \| f - p \|, \tag{1}$$

then $\Omega(f)$ consists of a single element.

A good part of [3] is devoted to the proof of this theorem by the same methods and ideas as those used in the proofs of unicity in Monotone Approximation [2] and Restricted Derivatives Approximation [1, 4]. This treatment can be avoided in view of the following observation:

Let $d = \| f - p \|_F$, $p \in \Omega(f)$, and let

$$K = \{h \mid h \in \Pi_n, f^{(k_i)}(x) - d \leqslant h^{(k_i)}(x) \leqslant f^{(k_i)}(x) + d, \quad a \leqslant x \leqslant b, i = 0, 1, 2, \ldots, p\}. \tag{2}$$

Obviously, $K = \Omega(f)$, and by assumption (1), $\| f - h \| = d$ for any $h \in K$. Therefore, the uniqueness of the polynomial of best approximation to f from K in the supremum norm [1] implies that K, i.e., $\Omega(f)$, contains exactly one element.

The uniqueness of RDA as proved in [4] cannot be applied to K since K does not satisfy two of the conditions assumed there [4, p. 217, lines 5, 6, and p. 222, assumption (viii)]. Yet, a more refined version of the same proof yields unicity without the above assumptions [1].

Since the treatment in [1] is carried out in a general setting, it is worthwhile to sketch the modification of the proof in [4]. In the following, all notations and numbers pertain to [4].

The uniqueness in [4] depends on the assumption stated in p. 217, lines 5, 6, through the use of Theorem 2 in the proof of Lemma 2. This can be avoided if in Lemma 2 the polynomial $Q \in \Pi_{k_0-1}$ (p. 225) is constructed to satisfy (13), but with lines 2 and 3 in (13) replaced by conditions (4) for $k_i < k_0$. By continuity arguments, for $\lambda > 0$ but small enough, P and λQ satisfy (2) and fail to satisfy (1). Thus the needed contradiction in Lemma 2 follows directly from Theorem 1.

The second assumption (4, p. 222, (viii)) can be omitted by a slight change in the construction of the incidence matrix E (p. 224). In case of overlapping pairs in conditions (a)–(e) (p. 224) (which cannot occur if (viii) is assumed), only the maximal even number of units corresponding to consecutive conditions at a point are introduced to the matrix E.

References

1. B. L. Chalmers, A unified approach to uniform real approximation by polynomials with linear restrictions, *Trans. Amer. Math. Soc.* **166** (1972), 309–316.
2. R. A. Lorentz, Uniqueness of approximation by monotone polynomials, *J. Approximation Theory* **4** (1971), 401–418.
3. R. A. Lorentz, Nonuniqueness of simultaneous approximation by algebraic polynomials, *J. Approximation Theory* **13** (1975), 17–23.
4. J. A. Roulier and G. D. Taylor, Approximation by polynomials with restricted ranges of their derivatives, *J. Approximation Theory* **5** (1972), 216–227.

E. Kimchi and N. Richter-Dyn

Department of Mathematical Sciences
Tel-Aviv University, Tel-Aviv, Israel
Communicated by Oved Shisha
Received May 5, 1975